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2000 J. Phys. A: Math. Gen. 33 6187

(http://iopscience.iop.org/0305-4470/33/35/307)

J. Phys. A: Math. Gen. **33** (2000) 6187–6202. Printed in the UK PII: S0305-4470(00)10710-3

Boundary impurities in the generalized supersymmetric *t–J* model

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Received 5 January 2000, in final form 15 June 2000

Abstract. We study the generalized supersymmetric t–J model with impurities in the boundaries. We first construct the higher spin operator K-matrix for the *XXZ* Heisenberg chain. Setting the boundary parameter to be a special value, we find a higher spin reflecting K-matrix for the supersymmetric t–J model. By using the quantum inverse scattering method, we obtain the eigenvalue and the corresponding Bethe ansatz equations.

1. Introduction

There has been extensive interest in the investigation of low-dimensional correlated electron systems with impurities. Recently, using renormalization group techniques, Kane and Fisher [1] studied the transport properties of a one-dimensional (1D) interacting electron gas in the presence of a potential barrier. They showed that a single potential scatter may dramatically influence the physics in the presence of repulsive e–e interactions. The system behaves like a Tomonaga–Luttinger liquid rather than a Fermi liquid. Some different techniques were also applied to study similar systems [2, 3]. The Kondo impurities in a Tomonaga–Luttinger liquid have been investigated in great detail [4–7].

Attempts to study the effects due to the presence of impurities in 1D quantum chains within the framework of integrable models have a long successful history [8–13]. Andrei and Johannesson [9] studied an arbitrary spin S embedded in a spin-$\frac{1}{2}$ Heisenberg chain. This method was generalized to other cases. Recently, the supersymmetric t–J model with impurities has attracted considerable interests. The Hamiltonian of the t–J model includes the near-neighbour hopping (t) and antiferromagnetic exchange (J) [16, 17]

$$H=\sum_{j=1}^{L}\left\{-t\mathcal{P}\sum_{\sigma=\pm1}(c_{j,\sigma}^{\dagger}c_{j+1,\sigma}+c_{j+1,\sigma}^{\dagger}c_{j,\sigma})\mathcal{P}+J\left(\boldsymbol{S}_j\boldsymbol{S}_{j+1}-\frac{1}{4}n_nn_{j+1}\right)\right\}. \tag{1}$$

It is known that this model is supersymmetric and integrable for $J=\pm2t$ [18, 19]. The supersymmetric t–J model was also studied in [20–23], for a review, see [24] and the references therein. Essler and Korepin [23] showed that the one-dimensional Hamiltonian can be obtained from the transfer matrix of the two-dimensional supersymmetric exactly solvable lattice model [22, 25].

By use of the quantum inverse scattering method (QISM) [26], the supersymmetric t–J model with higher spin impurity was first investigated in the periodic boundary conditions

[15]. Recently, the supersymmetric t–J model with impurities have been studied extensively in both periodic and reflecting (open) boundary conditions [27–31].

The open boundary condition has been studied extensively over the last decade. There have been several methods to study the problem of the open boundary condition [32, 33]. At the end of 1980s, Sklyanin [34] proposed a systematic approach to handle the open boundary condition problem within the framework of the QISM. Besides the Yang–Baxter equation [35], the reflection equation proposed by Cherednik [36] also plays a key role in proving the commutativity of the transfer matrix. We know that the Hamiltonian of the model is usually written as the logarithmic derivative of a transfer matrix at zero spectral parameter. The boundary terms in the Hamiltonian are determined by the reflecting K-matrix which is a solution to the reflection equation. In the usual boundary problem, the K-matrix is a c-number matrix. The operator K-matrices which determine the impurities in the Hamiltonian have been studied recently for several models [37], including the supersymmetric t–J model [27, 28]. In this paper, we will study the operator K-matrices for the generalized (q-deformed) supersymmetric t–J model and find the eigenvalues of the corresponding transfer matrix.

The Hamiltonian (1) of the supersymmetric t–J model can be obtained from the transfer matrix constructed by the rational R-matrix. We can also use the trigonometric R-matrix to formulate the transfer matrix. The corresponding Hamiltonian is a generalization of the original supersymmetric t–J model [38]. This Hamiltonian satisfies a symmetry of the quantum group $SU_q(2|1)$. In this paper, we shall study the generalized supersymmetric t–J model with higher spin boundary impurities. The operator K-matrix is first constructed for the *XXZ* Heisenberg spin chain with higher spin impurities. We then find a higher spin operator K-matrix for the supersymmetric t–J model. Using the graded algebraic Bethe ansatz method, We obtain the eigenvalue of the transfer matrix and the Bethe ansatz equations.

The paper is organized as follows. We introduce the model in section 2. In section 3, we study the *XXZ* spin chain with higher spin impurities and present the higher spin reflecting matrices for the generalized supersymmetric t–J model. In section 4, using the nested algebraic Bethe ansatz method, we obtain the eigenvalues of the transfer matrix for the generalized supersymmetric t–J model. Section 5 includes a brief summary and discussions.

2. The model

We first review the generalized supersymmetric t–J model. For convenience, we choose a similar notation to that in [23] and our previous paper [39]. The Hamiltonian of the generalized supersymmetric t–J model takes the following form:

$$
\begin{aligned}
H = \sum_{j=1}^{N}\sum_{\sigma=\pm}&[c_{j,\sigma}^{\dagger}(1-n_{j,-\sigma})c_{j+1,\sigma}(1-n_{j+1,-\sigma}) + c_{j+1,\sigma}^{\dagger}(1-n_{j+1,-\sigma})c_{j,\sigma}(1-n_{j,-\sigma})] \\
&-2\sum_{j=1}^{N}\left[\frac{1}{2}(S_j^{\dagger}S_{j+1}+S_jS_{j+1}^{\dagger})+\cos(\eta)S_j^zS_{j+1}^z-\frac{\cos(\eta)}{4}n_jn_{j+1}\right] \\
&+\mathrm{i}\sin(\eta)\sum_{j=1}^{N}[S_j^zn_{j+1}-S_{j+1}^zn_j].
\end{aligned}
\tag{2}
$$

When the anisotropic parameter $\eta = 0$, this Hamiltonian reduces to an equivalent form of the original Hamiltonian (1). The operators $c_{j,\sigma}$ and $c_{j,\sigma}^{\dagger}$ mean the annihilation and creation operators of an electron with spin σ on a lattice site j, and we assume the total number of lattice sites is N, with $\sigma = \pm 1$ representing spin down and up, respectively. These operators are canonical Fermi operators satisfying anticommutation relations

$$\{c_{j,\sigma}^{\dagger}, c_{j,\tau}\} = \delta_{ij}\delta_{\sigma\tau}. \tag{3}$$

We denote by $n_{j,\sigma} = c_{j,\sigma}^{\dagger} c_{j,\sigma}$ the number operator for the electron on a site j with spin σ, and by $n_j = \sum_{\sigma=\pm} n_{j,\sigma}$ the number operator for the electron on a site j. The Fock vacuum state $|0\rangle$ is defined as $c_{j,\sigma}|0\rangle = 0$. Due to the exclusion of double occupancy, there are altogether three possible electronic states at a given lattice site j,

$$|0\rangle \qquad |\uparrow\rangle_j = c_{j,1}^{\dagger}|0\rangle \qquad |\downarrow\rangle_j = c_{j,-1}^{\dagger}|0\rangle. \tag{4}$$

S_j^z, S_j, $S_j^{\dagger}$ are spin operators satisfying the $su(2)$ algebra and can be expressed as

$$S_j = c_{j,1}^{\dagger} c_{j,-1} \qquad S_j^{\dagger} = c_{j,-1}^{\dagger} c_j \qquad S_j^z = \frac{1}{2}(n_{j,1} - n_{j,-1}). \tag{5}$$

The above Hamiltonian can be obtained from the logarithmic derivative of the transfer matrix at zero spectral parameter. Within the framework of QISM, the transfer matrix is constructed by the trigonometric R-matrix of the Perk–Schultz model [40]. The non-zero entries of the R-matrix are given by

$$\begin{aligned} &\tilde{R}(\lambda)_{aa}^{aa} = \sin(\eta + \epsilon_a \lambda) \\ &\tilde{R}(\lambda)_{ab}^{ab} = (-1)^{\epsilon_a \epsilon_b} \sin(\lambda) \\ &\tilde{R}(\lambda)_{ba}^{ab} = \mathrm{e}^{\mathrm{i}\,\mathrm{sign}(a-b)\lambda} \sin(\eta) \qquad a \neq b \end{aligned} \tag{6}$$

where ϵ_a is the Grassman parity, $\epsilon_a = 0$ for boson and $\epsilon_a = 1$ for fermion, and

$$\mathrm{sign}(a-b) = \begin{cases} 1 & \text{if} \quad a > b \\ -1 & \text{if} \quad a < b. \end{cases} \tag{7}$$

This R-matrix of the Perk–Schultz model satisfies the usual Yang–Baxter equation:

$$\tilde{R}_{12}(\lambda - \mu)\tilde{R}_{13}(\lambda)\tilde{R}_{23}(\mu) = \tilde{R}_{23}(\mu)\tilde{R}_{13}(\lambda)\tilde{R}_{12}(\lambda - \mu). \tag{8}$$

In this paper, we shall concentrate our discussion only to the fermionic, fermionic and bosonic case (FFB), that means $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = 0$. And we shall use the graded formulae to study this model. For the supersymmetric t–J model, the spin of the electrons and the charge 'hole' degrees of freedom play a very similar role forming a graded superalgebra with two fermions and one boson. The holes obey boson commutation relations, while the spinors are fermions [24]. The graded approach has an advantage of making clear the distinction between bosonic and fermionic degrees of freedom [41].

Introducing a diagonal matrix $\Pi_{ac}^{bd} = (-)^{\epsilon_a \epsilon_c} \delta_{ab}\delta_{cd}$, we change the original R-matrix to the following form:

$$R(\lambda) = \Pi \tilde{R}(\lambda). \tag{9}$$

From the non-zero elements of the R-matrix R_{ab}^{cd}, we see that $\epsilon_a + \epsilon_b + \epsilon_c + \epsilon_d = 0$. One can show that the R-matrix satisfies the graded Yang–Baxter equation

$$R(\lambda - \mu)_{a_1 a_2}^{b_1 b_2} R(\lambda)_{b_1 a_3}^{c_1 b_3} R(\mu)_{b_2 b_3}^{c_2 c_3} (-)^{(\epsilon_{b_1} + \epsilon_{c_1})\epsilon_{b_2}} = R(\mu)_{a_2 a_3}^{b_2 b_3} R(\lambda)_{a_1 b_3}^{b_1 c_3} R(\lambda - \mu)_{b_1 b_2}^{c_1 c_2} (-)^{(\epsilon_{a_1} + \epsilon_{b_1})\epsilon_{b_2}}. \tag{10}$$

Explicitly, the R-matrix is written as

$$R(\lambda)=\begin{pmatrix} a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & b(\lambda) & 0 & -c_-(\lambda) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & b(\lambda) & 0 & 0 & 0 & c_-(\lambda) & 0 & 0\\ 0 & -c_+(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a(\lambda) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & c_-(\lambda) & 0\\ 0 & 0 & c_+(\lambda) & 0 & 0 & 0 & b(\lambda) & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & c_+(\lambda) & 0 & b(\lambda) & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w(\lambda) \end{pmatrix} \tag{11}$$

where

$$a(\lambda)=\sin(\lambda-\eta) \qquad w(\lambda)=\sin(\lambda+\eta) \qquad b(\lambda)=\sin(\lambda) \qquad c_\pm(\lambda)=\mathrm{e}^{\pm\mathrm{i}\lambda}\sin(\eta). \tag{12}$$

Within the framework of the QISM, we can construct the L operator from the R-matrix as

$$L_n(\lambda)=\begin{pmatrix} b(\lambda)-(b(\lambda)-a(\lambda))e_{11}^n & -c_-(\lambda)e_{21}^n & c_-(\lambda)e_{31}^n\\ -c_+(\lambda)e_{12}^n & b(\lambda)-(b(\lambda)-a(\lambda))e_{22}^n & c_-(\lambda)e_{32}^n\\ c_+(\lambda)e_{13}^n & c_+(\lambda)e_{23}^n & b(\lambda)-(b(\lambda)-w(\lambda))e_{33}^n \end{pmatrix}. \tag{13}$$

Here e_{ab}^n acts on the nth quantum space. Thus we have the (graded) Yang–Baxter relation

$$R_{12}(\lambda-\mu)L_1(\lambda)L_2(\mu)=L_2(\mu)L_1(\lambda)R_{12}(\lambda-\mu). \tag{14}$$

Here the tensor product is in the sense of super-tensor product defined as

$$(F\otimes G)_{ac}^{bd}=F_a^bG_c^d(-)^{(\epsilon_a+\epsilon_b)\epsilon_c}. \tag{15}$$

Except in section 3.1, all tensor products in this paper are in the sense of super-tensor products.

The row-to-row monodromy matrix $T_N(\lambda)$ is defined as a matrix product over the N operators on all sites of the lattice,

$$T_a(\lambda)=L_{aN}(\lambda)L_{aN-1}(\lambda)\cdots L_{a1}(\lambda) \tag{16}$$

where the subscript a represents the auxiliary space, $1,\ldots,N$ represent the quantum spaces in which the tensor product is in the graded sense. Explicitly, we write [23]

$$\{[T(\lambda)]^{ab}\}_{\beta_1\cdots\beta_N}^{\alpha_1\cdots\alpha_N}=L_N(\lambda)_{a\alpha_N}^{c_N\beta_N}L_{N-1}(\lambda)_{c_N\alpha_{N-1}}^{c_{N-1}\beta_{N-1}}\cdots L_1(\lambda)_{c_2\alpha_1}^{b\beta_1}(-1)^{\sum_{j=2}^N(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=1}^{j-1}\epsilon_{\alpha_i}}. \tag{17}$$

This definition is different from the non-graded case because there we have the graded Yang–Baxter equation (10). By repeatedly using the Yang–Baxter relation (14), one can prove easily that the monodromy matrix also satisfies the Yang–Baxter relation

$$R(\lambda-\mu)T_1(\lambda)T_2(\mu)=T_2(\mu)T_1(\lambda)R(\lambda-\mu). \tag{18}$$

For a periodic boundary condition, the transfer matrix $\tau_{peri}(\lambda)$ of this model is defined as the supertrace of the monodromy matrix in the auxiliary space

$$\tau_{peri}(\lambda)=\mathrm{str}\,T(\lambda)=\sum(-1)^{\epsilon_a}T(\lambda)_{aa}. \tag{19}$$

As a consequence of the Yang–Baxter relation (18) and the unitarity property of the R-matrix, we can prove that the transfer matrix commutes with each other for different spectral parameters,

$$[\tau_{peri}(\lambda), \tau_{peri}(\mu)] = 0. \tag{20}$$

In this sense we say that the model is integrable. Expanding the transfer matrix in powers of λ, we can find conserved quantities. And the Hamiltonian is defined as

$$H = \sin(\eta) \frac{\mathrm{d} \ln[\tau(\lambda)]}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = \sum_{j=1}^{N} H_{j,j+1} = \sum_{j=1}^{N} P_{j,j+1} L'_{j,j+1}(0) \tag{21}$$

where P_{ij} is the graded permutation operator expressed as $P_{ac}^{bd} = \delta_{ad}\delta_{bc}(-1)^{\epsilon_a\epsilon_c}$. The explicit expression of the Hamiltonian is presented in equation (2).

In this paper, we consider the reflecting boundary condition case. In addition to the Yang–Baxter equation, a reflection equation should be used in proving the commutativity of the transfer matrix with boundaries. The reflection equation takes the form [36]

$$R_{12}(\lambda-\mu)K_1(\lambda)R_{21}(\lambda+\mu)K_2(\mu) = K_2(\mu)R_{12}(\lambda+\mu)K_1(\lambda)R_{21}(\lambda-\mu). \tag{22}$$

For the graded case, the reflection equation remains the same as the above form. We only need to change the usual tensor product to the graded tensor product. We write it explicitly as

$$\begin{aligned} R(\lambda-\mu)_{a_1a_2}^{b_1b_2} K(\lambda)_{b_1}^{c_1} R(\lambda+\mu)_{b_2c_1}^{c_2d_1} K(\mu)_{c_2}^{d_2} (-)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}} \\ = K(\mu)_{a_2}^{b_2} R(\lambda+\mu)_{a_1b_2}^{b_1c_2} K(\lambda)_{b_1}^{c_1} R(\lambda-\mu)_{c_2c_1}^{d_2d_1} (-)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{c_2}}. \end{aligned} \tag{23}$$

Instead of the monodromy matrix $T(\lambda)$ for periodic boundary conditions, we consider the double-row monodromy matrix

$$\mathcal{T}(\lambda) = T(\lambda)K(\lambda)T^{-1}(-\lambda) \tag{24}$$

for the reflecting boundary conditions. Using the Yang–Baxter relation, and considering the boundary K-matrix which satisfies the reflection equation, one can prove that the double-row monodromy matrix $\mathcal{T}(\lambda)$ also satisfies the reflection equation

$$\begin{aligned} R(\lambda-\mu)_{a_1a_2}^{b_1b_2} \mathcal{T}(\lambda)_{b_1}^{c_1} R(\lambda+\mu)_{b_2c_1}^{c_2d_1} \mathcal{T}(\mu)_{c_2}^{d_2} (-)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}} \\ = \mathcal{T}(\mu)_{a_2}^{b_2} R(\lambda+\mu)_{a_1b_2}^{b_1c_2} \mathcal{T}(\lambda)_{b_1}^{c_1} R(\lambda-\mu)_{c_2c_1}^{d_2d_1} (-)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{c_2}}. \end{aligned} \tag{25}$$

Next, we study the properties of the R-matrix. We define the super-transposition st as

$$(A^{st})_{ij} = A_{ji}(-1)^{(\epsilon_i+1)\epsilon_j}. \tag{26}$$

For FFB grading used in this paper, $\epsilon_1 = \epsilon_2 = 1, \epsilon_3 = 0$, we can rewrite the above relation explicitly as

$$\begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{pmatrix}^{st} = \begin{pmatrix} A_{11} & A_{21} & C_1 \\ A_{12} & A_{22} & C_2 \\ -B_1 & -B_2 & D \end{pmatrix}. \tag{27}$$

We also define the inverse of the super-transposition $\bar{st}$ as $\{A^{st}\}^{\bar{st}} = A$.

One can prove directly that the R-matrix (11) satisfies the following unitarity and cross-unitarity relations:

$$R_{12}(\lambda)R_{21}(-\lambda) = \rho(\lambda)\cdot id \qquad \rho(\lambda) = \sin(\eta+\lambda)\sin(\eta-\lambda) \tag{28}$$

$$R_{12}^{st_1}(\eta-\lambda)M_1R_{21}^{st_1}(\lambda)M_1^{-1} = \tilde{\rho}(\lambda)\cdot id \qquad \tilde{\rho}(\lambda) = \sin(\lambda)\sin(\eta-\lambda). \tag{29}$$

Here the matrix $M = \mathrm{diag}(\mathrm{e}^{2\mathrm{i}\eta}, 1, 1)$ is determined by the R-matrix. The cross-unitarity relation can also be written as the following form:

$$\left\{M_1^{-1} R_{12}^{st_1 st_2}(\eta - \lambda) M_1\right\}^{\bar{st}_2} R_{21}^{st_1}(\lambda) = \tilde{\rho}(\lambda) \tag{30}$$

$$R_{12}^{st_1}(\lambda) \left\{M_1 R_{21}^{st_1 st_2}(\eta - \lambda) M_1^{-1}\right\}^{\bar{st}_2} = \tilde{\rho}(\lambda). \tag{31}$$

In order to construct the commuting transfer matrix with boundaries, besides the reflection equation, we need the dual reflection equation. In general, the dual reflection equation which depends on the unitarity and cross-unitarity relations of the R-matrix takes different forms for different models. For the models considered in this paper, we can write the dual reflection equation in the following form:

$$\begin{aligned} &R_{21}^{st_1 st_2}(\mu - \lambda) K_1^{+st_1}(\lambda) M_1^{-1} R_{12}^{st_1 st_2}(\eta - \lambda - \mu) M_1 K_2^{+st_2}(\mu) \\ &\quad = K_2^{+st_2}(\mu) M_1 R_{21}^{st_1 st_2}(\eta - \lambda - \mu) M_1^{-1} K_1^{+st_1}(\lambda) R_{12}^{st_1 st_2}(\mu - \lambda). \end{aligned} \tag{32}$$

Then the transfer matrix with boundaries is defined as

$$t(\lambda) = \mathrm{str}\, K^+(\lambda) \mathcal{T}(\lambda). \tag{33}$$

The commutativity of $t(\lambda)$ can be proved by using unitarity and cross-unitarity relations, reflection equation and the dual reflection equation. The detailed proof of the commuting transfer matrix with boundaries for super (graded) case can be found, for instance, in [42–44] etc. With a normalization $K(0) = id$, the Hamiltonian can be obtained as

$$\begin{aligned} H &\equiv \frac{1}{2} \sin(\eta) \frac{\mathrm{d} \ln t(\lambda)}{\mathrm{d}\lambda} \bigg|_{\lambda=0} \\ &= \sum_{j=1}^{N-1} P_{j,j+1} L'_{j,j+1}(0) + \frac{1}{2} \sin(\eta) K'_1(0) + \frac{\mathrm{str}_a\, K_a^+(0) P_{Na} L'_{Na}(0)}{\mathrm{str}_a\, K_a^+(0)}. \end{aligned} \tag{34}$$

3. Higher spin solution to the reflection equation for supersymmetric *t–J* model

In order to find the higher spin solution to the reflection equation for the generalized supersymmetric *t–J* model, we first construct the higher spin reflecting matrix for the *XXZ* Heisenberg chain.

3.1. XXZ Heisenberg chain with higher spin boundary impurities

The higher spin R-matrix can be constructed by using the fusion procedure [45]. The Hamiltonian of the *XXZ* Heisenberg chain is written as

$$H_{XXZ} = \sum_{j=1}^{N} \left[\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \frac{1}{2} \cos(\eta) \sigma_j^z \sigma_{j+1}^z\right]. \tag{35}$$

Here $\sigma^{\pm} = 1/2(\sigma^x \pm \sigma^y)$ and σ^x, σ^y and σ^z are Pauli matrices. The R-matrix is known to be the standard six-vertex model,

$$r_{12}(\lambda) = \begin{pmatrix} \sin(\lambda + \eta) & 0 & 0 & 0 \\ 0 & \sin(\lambda) & \sin(\eta) & 0 \\ 0 & \sin(\eta) & \sin(\lambda) & 0 \\ 0 & 0 & 0 & \sin(\lambda + \eta) \end{pmatrix}. \tag{36}$$

Within the framework of QISM, the L operator constructed by the R-matrix is written as

$$L_{ak}(\lambda) = \begin{pmatrix} \sin(\lambda + \frac{1}{2}\eta + \frac{1}{2}\eta\sigma_k^z) & \sin(\eta)\sigma_k^- \\ \sin(\eta)\sigma_k^+ & \sin(\lambda + \frac{1}{2}\eta - \frac{1}{2}\eta\sigma_k^z) \end{pmatrix} \tag{37}$$

where a represents auxiliary space. As usual, we can construct the row-to-row monodromy matrix $T_a(\lambda) = L_{aN}(\lambda)\cdots L_{a1}(\lambda)$, and we have the Yang–Baxter relation

$$r_{12}(\lambda - \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)r_{12}(\lambda - \mu) \tag{38}$$

where the tensor product is a non-graded one.

Next, we consider the higher spin operators. Let the higher spin L operator take the form [45, 46]

$$\mathcal{L}(\lambda) = \begin{pmatrix} \sin(\lambda + \boldsymbol{S}^z\eta) & \sin(\eta)\boldsymbol{S}^- \\ \sin(\eta)\boldsymbol{S}^+ & \sin(\lambda - \boldsymbol{S}^z\eta) \end{pmatrix} \tag{39}$$

where $\boldsymbol{S}^z$, $\boldsymbol{S}$ and $\boldsymbol{S}^\dagger$ are spin-s operators satisfying the following commutation relations:

$$[\boldsymbol{S}^z, \boldsymbol{S}^\pm] = \pm\boldsymbol{S}^\pm \qquad [\boldsymbol{S}^+, \boldsymbol{S}^-] = \frac{\sin(2\boldsymbol{S}^z\eta)}{\sin(\eta)}. \tag{40}$$

We also have the following relations for the spin-s operator:

$$\begin{aligned} &\sin(\boldsymbol{S}^z\eta)\sin(\eta + \boldsymbol{S}^z\eta) + \sin^2(\eta)\boldsymbol{S}^-\boldsymbol{S}^+ \\ &\quad = \sin^2(\eta)\boldsymbol{S}^+\boldsymbol{S}^- + \sin(\boldsymbol{S}^z\eta)\sin(\boldsymbol{S}^z\eta - \eta) = \sin(s\eta)\sin(s\eta + \eta). \end{aligned} \tag{41}$$

A more general relation can be written as

$$\begin{aligned} &\sin(\lambda + \boldsymbol{S}^z\eta)\sin(\eta + \boldsymbol{S}^z\eta - \lambda) + \sin^2(\eta)\boldsymbol{S}^-\boldsymbol{S}^+ \\ &\quad = \sin^2(\eta)\boldsymbol{S}^+\boldsymbol{S}^- + \sin(\lambda - \boldsymbol{S}^z\eta)\sin(-\lambda - \boldsymbol{S}^z\eta + \eta) \\ &\quad = \sin(\lambda + s\eta)\sin(s\eta + \eta - \lambda). \end{aligned} \tag{42}$$

One can prove that the higher spin L operator also satisfies the Yang–Baxter relation

$$r_{12}(\lambda - \mu)\mathcal{L}_1(\lambda)\mathcal{L}_2(\mu) = \mathcal{L}_2(\mu)\mathcal{L}_1(\lambda)r_{12}(\lambda - \mu). \tag{43}$$

Now, let us consider the reflecting boundary condition. We can find a c-number solution to the reflection equation $K_c(\lambda) = \mathrm{diag}\,(\sin(\xi + \lambda), \sin(\xi - \lambda))$, where ξ is an arbitrary parameter. This is a general c-number diagonal solution to the reflection equation. In particular, if $\xi \to -\mathrm{i}\infty$, we find $K(\lambda) = \mathrm{diag}(\mathrm{e}^{2\mathrm{i}\lambda}, 1)$ is a solution to the reflection equation.

It is interesting to find a higher spin operator K-matrix. We can construct the operator K-matrix by $\mathcal{K}_{XXZ}(\lambda) = \mathcal{L}(\lambda + c)K_c(\lambda)\mathcal{L}^{-1}(-\lambda + c)$, one can find easily that $\mathcal{K}(\lambda)$ is an operator reflecting matrix satisfying the reflection equation. Explicitly, the higher spin reflecting $\mathcal{K}$ has the form

$$\mathcal{K}_{XXZ}(\lambda) = \begin{pmatrix} K(\lambda)_1^1 & K(\lambda)_1^2 \\ K(\lambda)_2^1 & K(\lambda)_2^2 \end{pmatrix}$$

with

$$\begin{aligned} K(\lambda)_1^1 &= \sin(\lambda - \xi)\sin(\lambda + c + s\eta)\sin(\lambda + c - \eta - s\eta) \\ &\qquad + \sin(2\lambda)\sin(\lambda + c + \boldsymbol{S}^z\eta)\sin(\xi - c + \eta + \boldsymbol{S}^z\eta) \\ K(\lambda)_2^2 &= -\sin(\xi + \lambda)\sin(\lambda + c + s\eta)\sin(\lambda + c - \eta - s\eta) \\ &\qquad + \sin(2\lambda)\sin(\lambda + c - \boldsymbol{S}^z\eta)\sin(\xi + c - \eta + \boldsymbol{S}^z\eta) \\ K(\lambda)_1^2 &= \sin(\eta)\sin(2\lambda)\sin(\xi + c + \boldsymbol{S}^z\eta)\boldsymbol{S}^- \\ K(\lambda)_2^1 &= \sin(\eta)\sin(2\lambda)\sin(\xi - c + \boldsymbol{S}^z\eta)\boldsymbol{S}^+. \end{aligned} \tag{44}$$

By use of the cross-unitarity relation of the R-matrix, the operator reflecting matrix to the dual reflection equation can also be found. The eigenvalues of the transfer matrix can be obtained by applying the algebraic Bethe ansatz method.

3.2. Higher spin reflecting matrix for the supersymmetric t–J model

We know that the generalized supersymmetric t–J model has an $SU_q(2)$ symmetry. We suppose that the operator K-matrix takes the following form:

$$K(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) & 0 \\ B(\lambda) & C(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{45}$$

Inserting this matrix into the reflection equation (23), we can find the following non-trivial relations:

$$\hat{r}(\lambda-\mu)^{b_1b_2}_{a_1a_2}K(\lambda)^{c_1}_{b_1}\hat{r}(\lambda+\mu)^{c_2d_1}_{b_2c_1}K(\mu)^{d_2}_{c_2} = K(\mu)^{b_2}_{a_2}\hat{r}(\lambda+\mu)^{b_1c_2}_{a_1b_2}K(\lambda)^{c_1}_{b_1}\hat{r}(\lambda-\mu)^{d_2d_1}_{c_2c_1} \tag{46}$$

and

$$K(\lambda)^{b_1}_{a_1}K(\mu)^{d_1}_{b_1} = K(\mu)^{b_1}_{a_1}K(\lambda)^{d_1}_{b_1} \tag{47}$$

$$\begin{aligned}\delta_{a_1d_1}\sin(\lambda-\mu)\,\mathrm{e}^{-\mathrm{i}(\lambda+\mu)} + \sin(\lambda+\mu)\,\mathrm{e}^{\mathrm{i}(\lambda-\mu)}K(\lambda)^{d_1}_{a_1}& \\ = \mathrm{e}^{-\mathrm{i}(\lambda-\mu)}\sin(\lambda+\mu)K(\mu)^{d_1}_{a_1} + \mathrm{e}^{\mathrm{i}(\lambda+\mu)}K(\mu)^{b_1}_{a_1}K(\lambda)^{d_1}_{b_1}&\end{aligned} \tag{48}$$

where all indices take values 1, 2, and we have introduced

$$\hat{r}_{12}(\lambda) = \begin{pmatrix} \sin(\lambda-\eta) & 0 & 0 & 0 \\ 0 & \sin(\lambda) & -\sin(\eta)\,\mathrm{e}^{-\mathrm{i}\lambda} & 0 \\ 0 & -\sin(\eta)\,\mathrm{e}^{\mathrm{i}\lambda} & \sin(\lambda) & 0 \\ 0 & 0 & 0 & \sin(\lambda-\eta) \end{pmatrix}. \tag{49}$$

This matrix $\hat{r}(\lambda)$ can be obtained from the matrix (36) by a gauge transformation and with a change $\eta \to -\eta$. Correspondingly, we can show that $A(\lambda) = f(\lambda)\mathrm{e}^{-2\mathrm{i}\lambda}K(\lambda)^1_1$, $B(\lambda) = f(\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda}K(\lambda)^2_1$, $C(\lambda) = f(\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda}K(\lambda)^1_2$, $D(\lambda) = f(\lambda)K(\lambda)^2_1$ satisfy relation (46). Substituting these results into relations (47) and (48), and after some tedious calculations, we find that if we take $\xi \to -\mathrm{i}\infty$, and $f(\lambda) = -1/\mathrm{e}^{2\mathrm{i}\lambda}\sin(\lambda-c-\eta-s\eta)\sin(\lambda-c+s\eta)$, all relations obtained from the reflection equation can be satisfied. So, we finally find the higher spin reflecting matrix as

$$\begin{aligned} A(\lambda) &= g(\lambda)\big(\mathrm{e}^{-4\mathrm{i}\lambda}\sin(\lambda+c-s\eta)\sin(\lambda+c+\eta+s\eta) \\ &\qquad - \sin(2\lambda)\sin(\lambda+c-\boldsymbol{S}^z\eta)\,\mathrm{e}^{-\mathrm{i}(3\lambda+c+\eta+\boldsymbol{S}^z\eta)}\big) \\ B(\lambda) &= g(\lambda)\sin(\eta)\sin(2\lambda)\,\mathrm{e}^{-\mathrm{i}(2\lambda-c+\boldsymbol{S}^z\eta)}\boldsymbol{S}^- \\ C(\lambda) &= g(\lambda)\sin(\eta)\sin(2\lambda)\,\mathrm{e}^{-\mathrm{i}(2\lambda+c+\boldsymbol{S}^z\eta)}\boldsymbol{S}^+ \\ D(\lambda) &= g(\lambda)\left(\sin(\lambda+c-s\eta)\sin(\lambda+c+\eta+s\eta) - \sin(2\lambda)\sin(\lambda+c+\boldsymbol{S}^z\eta)\,\mathrm{e}^{-\mathrm{i}(\lambda-c-\eta+\boldsymbol{S}^z\eta)}\right) \end{aligned} \tag{50}$$

where $g(\lambda) = 1/\sin(\lambda-c-\eta-s\eta)\sin(\lambda-c+s\eta)$.

Next, let us consider the higher spin reflecting matrix to the dual reflection equation (32). We suppose K^+ has the similar form as K. By direct calculation, we can find $R^{st_1st2}_{12}(\lambda) =$

$I_1R_{21}(\lambda)I_1$ with $I = \mathrm{diag}(-1, -1, 1)$. For the form (45), we have $IK(\lambda)I = K(\lambda)$. Then with the help of property $[M_1M_2, R(\lambda)] = 0$, we can write the dual reflection equation as

$$R_{12}(\mu-\lambda)K_1^{+st_1}(\lambda)M_1^{-1}R_{21}(\eta-\lambda-\mu)K_2^{+st_2}(\mu)M_2^{-1}$$
$$= K_2^{+st_2}(\mu)M_2^{-1}R_{12}(\eta-\lambda-\mu)K_1^{+st_1}(\lambda)M_1^{-1}R_{21}(\mu-\lambda). \tag{51}$$

We see that there is an isomorphism between K and K^+:

$$K(\lambda) :\to K^{+st}(\lambda) = K\left(\frac{\eta}{2}-\lambda\right)M. \tag{52}$$

Given a solution to the reflection equation (23), we can also find a solution to the dual reflection equation (51). Remark that in the sense of the transfer matrix, the reflection equation and the dual reflection equation are independent of each other. We can write the higher spin reflecting matrix to the dual reflection equation as

$$K^+(\lambda) = \begin{pmatrix} A^+(\lambda) & B^+(\lambda) & 0 \\ B^+(\lambda) & C^+(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{53}$$

with

$$\begin{aligned} A^+(\lambda) &= g^+(\lambda)[\mathrm{e}^{4\mathrm{i}\lambda}\sin(\lambda+\tilde{c}-\eta+\tilde{s}\eta)\sin(\lambda+\tilde{c}-2\eta-\tilde{s}\eta). \\ &\qquad -\sin(2\lambda-\eta)\sin(\lambda+\tilde{c}-\eta-\tilde{\boldsymbol{S}}^z\eta)\,\mathrm{e}^{\mathrm{i}(3\lambda+\tilde{c}-\eta+\tilde{\boldsymbol{S}}^z\eta)}] \\ B^+(\lambda) &= -g^+(\lambda)\sin(\eta)\sin(2\lambda-\eta)\,\mathrm{e}^{\mathrm{i}(2\lambda+\tilde{c}+\frac{\eta}{2}+\tilde{\boldsymbol{S}}^z\eta)}\tilde{\boldsymbol{S}}^- \\ C^+(\lambda) &= -g^+(\lambda)\sin(\eta)\sin(2\lambda-\eta)\,\mathrm{e}^{\mathrm{i}(2\lambda-\tilde{c}-\frac{\eta}{2}+\tilde{\boldsymbol{S}}^z\eta)}\tilde{\boldsymbol{S}}^+ \\ D^+(\lambda) &= g^+(\lambda)[\sin(\lambda+\tilde{c}-\eta+\tilde{s}\eta)\sin(\lambda+\tilde{c}-2\eta-\tilde{s}\eta) \\ &\qquad -\sin(2\lambda-\eta)\sin(\lambda+\tilde{c}-\eta+\tilde{\boldsymbol{S}}^z\eta)\,\mathrm{e}^{\mathrm{i}(\lambda-\tilde{c}+\eta+\tilde{\boldsymbol{S}}^z\eta)}] \end{aligned} \tag{54}$$

where $g^+(\lambda) = 1/\sin(\lambda-\tilde{c}+\eta+\tilde{s}\eta)\sin(\lambda-\tilde{c}-\tilde{s}\eta)$.

Thus we find the higher spin reflecting matrices for the generalized supersymmetric t–J model. We should remark that these higher spin reflecting matrices are a kind of 'singular' matrices. They cannot be constructed directly by the Sklyanin's 'dressing' procedure. In the rational limit, they reduce to the result obtained in [28]. The rational higher spin K-matrix has been analysed in detail by the projecting method [30]. Our result should also be obtained by the projecting method.

By definition in equation (34), and using the explicit form of the boundary reflecting matrices (50) and (54), we can find the boundary impurity terms. The boundary impurity coupled to site 1 is written as

$$H_1 = \frac{2}{\sin(c+\eta+s\eta)\sin(c-s\eta)}\mathrm{e}^{-\mathrm{i}\boldsymbol{S}^z\eta}[\mathrm{e}^{\mathrm{i}c}\boldsymbol{S}^-S_1^\dagger + \mathrm{e}^{-\mathrm{i}c}\boldsymbol{S}^+S_1$$
$$+(\mathrm{e}^{-\mathrm{i}(c+\eta)}\sin(c-\boldsymbol{S}^z\eta)S_1^z - \mathrm{e}^{\mathrm{i}(c+\eta)}\sin(c+\boldsymbol{S}^z\eta)S_1^z)]. \tag{55}$$

The impurity coupled to site N is in a similar form.

4. Algebraic Bethe ansatz method for the generalized supersymmetric *t–J* model with higher spin impurities

4.1. First-level algebraic Bethe ansatz

We denote the double-row monodromy matrix as

$$\mathcal{T}(\lambda) = \begin{pmatrix} \mathcal{A}_{11}(\lambda) & \mathcal{A}_{12}(\lambda) & \mathcal{B}_1(\lambda) \\ \mathcal{A}_{21}(\lambda) & \mathcal{A}_{22}(\lambda) & \mathcal{B}_2(\lambda) \\ \mathcal{C}_1(\lambda) & \mathcal{C}_2(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}. \tag{56}$$

For later discussions, we introduce the following transformations:

$$\mathcal{A}_{ab}(\lambda) = \tilde{\mathcal{A}}_{ab}(\lambda) + \delta_{ab}\frac{\mathrm{e}^{-2\mathrm{i}\lambda}\sin(\eta)}{\sin(2\lambda+\eta)}\mathcal{D}(\lambda). \tag{57}$$

As mentioned in section 2, the double-row monodromy matrix satisfies the reflection equation (25), we have the following commutation relations:

$$\mathcal{C}_{d_1}(\lambda)\mathcal{C}_{d_2}(\mu) = -\frac{\hat{r}_{12}(\lambda-\mu)^{d_2d_1}_{c_2c_1}}{\sin(\lambda-\mu+\eta)}\mathcal{C}_{c_2}(\mu)\mathcal{C}_{c_1}(\lambda) \tag{58}$$

$$\begin{aligned}\mathcal{D}(\lambda)\mathcal{C}_d(\mu) = {}& \frac{\sin(\lambda+\mu)\sin(\lambda-\mu-\eta)}{\sin(\lambda+\mu+\eta)\sin(\lambda-\mu)}\mathcal{C}_d(\mu)\mathcal{D}(\lambda) \\ &+\frac{\sin(2\mu)\sin(\eta)\,\mathrm{e}^{\mathrm{i}(\lambda-\mu)}}{\sin(\lambda-\mu)\sin(2\mu+\eta)}\mathcal{C}_d(\lambda)\mathcal{D}(\mu) - \frac{\sin(\eta)\,\mathrm{e}^{\mathrm{i}(\lambda+\mu)}}{\sin(\lambda+\mu+\eta)}\mathcal{C}_b(\lambda)\tilde{\mathcal{A}}_{bd}(\mu)\end{aligned} \tag{59}$$

$$\begin{aligned}\tilde{\mathcal{A}}_{a_1d_1}(\lambda)\mathcal{C}_{d_2}(\mu) = {}& \frac{\hat{r}_{12}(\lambda+\mu+\eta)^{c_1b_2}_{a_1c_2}\hat{r}_{21}(\lambda-\mu)^{d_1d_2}_{b_1b_2}}{\sin(\lambda+\mu+\eta)\sin(\lambda-\mu)}\mathcal{C}_{c_2}(\mu)\tilde{\mathcal{A}}_{c_1b_1}(\lambda) \\ &+\frac{\sin(\eta)\,\mathrm{e}^{-\mathrm{i}(\lambda-\mu)}}{\sin(\lambda-\mu)\sin(2\lambda+\eta)}\hat{r}_{12}(2\lambda+\eta)^{b_2d_1}_{a_1b_1}\mathcal{C}_{b_1}(\lambda)\tilde{\mathcal{A}}_{b_2d_2}(\mu) \\ &-\frac{\sin(2\mu)\sin(\eta)\,\mathrm{e}^{-\mathrm{i}(\lambda+\mu)}}{\sin(\lambda+\mu+\eta)\sin(2\lambda+\eta)\sin(2\mu+\eta)}\hat{r}_{12}(2\lambda+\eta)^{d_2d_1}_{a_1b_2}\mathcal{C}_{b_2}(\lambda)\mathcal{D}(\mu).\end{aligned} \tag{60}$$

Here the indices take values 1, 2, and the matrix $\hat{r}$ is defined in (49).

We define a reference state in the nth quantum space as $|0\rangle_n = (0,0,1)^t$, and reference states for the boundary operators as $\boldsymbol{S}^-|0\rangle_r = 0$, $\boldsymbol{S}^z|0\rangle_r = -s|0\rangle_r$, $\boldsymbol{S}^+|0\rangle_r \neq 0$, and $\tilde{\boldsymbol{S}}^-|0\rangle_l = 0$, $\tilde{\boldsymbol{S}}^z|0\rangle_l = -\tilde{s}|0\rangle_l$, $\tilde{\boldsymbol{S}}^+|0\rangle_l \neq 0$. The vacuum state is then defined as $|0\rangle = |0\rangle_l \otimes_{k=1}^N |0\rangle_k \otimes |0\rangle_r$. Acting the double-row monodromy matrix on this vacuum state, we have

$$\begin{aligned}&\mathcal{B}_a(\lambda)|0\rangle = 0 \\ &\mathcal{C}_a(\lambda)|0\rangle \neq 0 \\ &\mathcal{D}(\lambda)|0\rangle = \sin^{2N}(\lambda+\eta)|0\rangle \\ &\tilde{\mathcal{A}}_{ab}(\lambda)|0\rangle = \sin^{2N}(\lambda)\left[K(\lambda)^b_a - \delta_{ab}\frac{\sin(\eta)\,\mathrm{e}^{-2\mathrm{i}\lambda}}{\sin(2\lambda+\eta)}\right]|0\rangle = W_{ab}(\lambda)\sin^{2N}(\lambda)|0\rangle\end{aligned} \tag{61}$$

where

$$
\begin{aligned}
&W_{12}(\lambda) = 0 \qquad W_{21}(\lambda) = C(\lambda)\\
&W_{11}(\lambda) = g(\lambda)\frac{\mathrm{e}^{\mathrm{i}\eta}\sin(2\lambda)}{\sin(2\lambda+\eta)}[\mathrm{e}^{-\mathrm{i}(4\lambda+2\eta)}\sin(\lambda+c+e-s\eta)\sin(\lambda+c+2\eta+s\eta)\\
&\qquad -\sin(2\lambda+\eta)\sin(\lambda+c+\eta+s\eta)\,\mathrm{e}^{-\mathrm{i}(3\lambda+c+3\eta-s\eta)}]\\
&W_{22}(\lambda) = -\mathrm{e}^{-2\mathrm{i}\lambda}\frac{\sin(2\lambda)\sin(\lambda+c+\eta-s\eta)}{\sin(2\lambda+\eta)\sin(\lambda-c+s\eta)}.
\end{aligned}
\tag{62}
$$

The transfer matrix (33) can be written as

$$
\begin{aligned}
t(\lambda) &= -K^{+}(\lambda)_{b}^{a}\mathcal{A}_{ab}(\lambda) + \mathcal{D}(\lambda)\\
&= -K^{+}(\lambda)_{a}^{b}\tilde{\mathcal{A}}_{ba}(\lambda) + \left(1 - \frac{\sin(\eta)\,\mathrm{e}^{-2\mathrm{i}\lambda}}{\sin(2\lambda+\eta)}[A^{+}(\lambda)+D^{+}(\lambda)]\right)\mathcal{D}(\lambda).
\end{aligned}
\tag{63}
$$

Acting this transfer matrix on the ansatz of the eigenvector

$$
\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}
\tag{64}
$$

where $F^{d_1\cdots d_n}$ is a function of the spectral parameters μ_j, we have

$$
\begin{aligned}
&t(\lambda)\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}\\
&\quad= \frac{\sin(2\lambda-\eta)\sin(\lambda-\tilde{c}+\eta-\tilde{s}\eta)\sin(\lambda-\tilde{c}+2\eta+\tilde{s}\eta)}{\sin(2\lambda+\eta)\sin(\lambda-\tilde{c}+\eta+\tilde{s}\eta)\sin(\lambda-\tilde{c}-\tilde{s}\eta)}\\
&\qquad\times\sin^{2N}(\lambda+\eta)\prod_{i=1}^{n}\frac{\sin(\lambda+\mu_i)\sin(\lambda-\mu_i-\eta)}{\sin(\lambda+\mu_i+\eta)\sin(\lambda-\mu_i)}\mathcal{C}_{d_1}(\mu_1)\cdots\mathcal{C}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}\\
&\qquad+\sin^{2N}(\lambda)\prod_{i=1}^{n}\frac{1}{\sin(\lambda-\mu_i)\sin(\lambda+\mu_i+\eta)}\mathcal{C}_{c_1}(\mu_1)\cdots\mathcal{C}_{c_n}(\mu_n)\\
&\qquad\times t^{(1)}(\lambda)_{d_1\cdots d_n}^{c_1\cdots c_n}|0\rangle F^{d_1\cdots d_n}\\
&\qquad+\text{u.t.}
\end{aligned}
\tag{65}
$$

where u.t. denotes unwanted terms, and $t^{(1)}(\lambda)$ is the so-called nested transfer matrix which can be defined, with the help of the relation (60), as

$$
\begin{aligned}
t^{(1)}(\lambda)_{d_1\cdots d_n}^{c_1\cdots c_n} &= -K^{+}(\lambda)_{b}^{a}\left\{\hat{r}(\lambda+\mu_1+\eta)_{ac_1}^{a_1e_1}\hat{r}(\lambda+\mu_2+\eta)_{a_1c_2}^{a_2e_2}\cdots\hat{r}(\lambda+\mu_1+\eta)_{a_{n-1}c_n}^{a_ne_n}\right\}\\
&\quad\times W_{a_nb_n}(\lambda)\left\{\hat{r}_{21}(\lambda-\mu_n)_{b_ne_n}^{b_{n-1}d_n}\cdots\hat{r}_{21}(\lambda-\mu_2)_{b_2e_2}^{b_1d_2}\hat{r}_{21}(\lambda-\mu_1)_{b_1e_1}^{bd_1}\right\}.
\end{aligned}
\tag{66}
$$

We find that this nested transfer matrix can be regarded as a transfer matrix with reflecting boundary conditions corresponding to the anisotropic case

$$
t^{(1)}(\lambda) = \operatorname{str} K^{(1)^{+}}(\lambda')T^{(1)}(\lambda',\{\mu_i'\})K^{(1)}(\lambda')T^{(1)^{-1}}(-\lambda',\{\mu_i'\})
\tag{67}
$$

with the grading $\epsilon_1 = \epsilon_2 = 1$. Here, we denote $\lambda' = \lambda + \frac{\eta}{2}$, $\mu' = \mu + \frac{\eta}{2}$. The reflecting matrix can also be interpreted as an operator matrix with higher spin. Explicitly, with the help of (62) and (63), we have

$$
\begin{aligned}
K^{(1)}(\lambda') &= \mathrm{e}^{\mathrm{i}\eta}\frac{\sin(2\lambda'-\eta)}{\sin(2\lambda')}\begin{pmatrix} A(\lambda',c') & B(\lambda',c')\\ C(\lambda',c') & D(\lambda',c')\end{pmatrix}\\
K^{(1)^{+}}(\lambda') &= \begin{pmatrix} A^{+}(\lambda'-\frac{\eta}{2}) & B^{+}(\lambda'-\frac{\eta}{2})\\ C^{+}(\lambda'-\frac{\eta}{2}) & D^{+}(\lambda'-\frac{\eta}{2})\end{pmatrix}
\end{aligned}
\tag{68}
$$

where $c' = c + \frac{\eta}{2}$. Note that the solution of the reflection equation can be changed by a gauge transformation. In order to prove that the above defined nested transfer matrix is still a transfer matrix with higher spin reflecting matrix, we should prove that $K^{(1)}(\lambda')$ and $K^{(1)^+}(\lambda')$ satisfy the reduced reflection equation and its corresponding dual reflection equation. Indeed, it can be shown that the following reflection equation holds:

$$\hat{r}_{12}(\lambda' - \mu')K_1^{(1)}(\lambda')\hat{r}_{21}(\lambda' + \mu')K_2^{(1)}(\mu') = K_2^{(1)}(\mu')\hat{r}_{12}(\lambda' + \mu')K_1^{(1)}(\lambda')\hat{r}_{21}(\lambda' - \mu'). \tag{69}$$

With $M^{(1)} = \mathrm{diag}(\mathrm{e}^{2\mathrm{i}\eta}, 1)$, and the isomorphism (52), we find that $K^{(1)^+}$ satisfies the following relation:

$$\begin{aligned}\hat{r}_{12}(-\lambda' + \mu')K^{(1)^+}{}_1(\lambda')^{st_1}{M^{(1)}}_1^{-1}\hat{r}_{21}(2\eta - \lambda' - \mu')K^{(1)^+}{}_2(\mu')^{st_2}{M^{(1)}}_2^{-1} \\ = K^{(1)^+}{}_2(\mu')^{st_2}{M^{(1)}}_2^{-1}\hat{r}_{12}(2\eta - \lambda' - \mu')K_1^{(1)}(\lambda'){M^{(1)}}_1^{-1}\hat{r}_{21}(-\lambda' + \mu').\end{aligned} \tag{70}$$

By use of the cross-unitarity relation $\hat{r}_{12}^{st_1}(2\eta - \lambda)M_1^{(1)}\hat{r}_{21}^{st_1}(\lambda){M_1^{(1)}}^{-1} = \sin(\lambda)\sin(2\eta - \lambda)\cdot id$, the above relation is just the dual reflection equation which we need.

The row-to-row monodromy matrix $T^{(1)}(\lambda', \{\mu_i'\})$ (corresponding to the periodic boundary condition) and its inverse are defined as

$$T_{aa_n}^{(1)}(\lambda', \{\mu_i'\})_{c_1\cdots c_n}^{e_1\cdots e_n} = \hat{r}(\lambda' + \mu_1')_{ac_1}^{a_1e_1}\hat{r}(\lambda' + \mu_2')_{a_1c_2}^{a_2e_2}\cdots\hat{r}(\lambda' + \mu_1')_{a_{n-1}c_n}^{a_ne_n} \tag{71}$$

$${T^{(1)}}_{b_na}^{-1}(-\lambda', \{\mu_i'\})_{e_n\cdots e_1}^{d_n\cdots d_1} = \hat{r}_{21}(\lambda' - \mu_n')_{b_ne_n}^{b_{n-1}d_n}\cdots\hat{r}_{21}(\lambda' - \mu_2')_{b_2e_2}^{b_1d_2}\hat{r}_{21}(\lambda' - \mu_1')_{b_1e_1}^{ad_1}. \tag{72}$$

We show that the problem of finding the eigenvalue of the original transfer matrix $t(\lambda)$ reduces to the problem of finding the eigenvalue of the nested transfer matrix $t^{(1)}(\lambda)$. The nested transfer matrix is still a boundary case with higher spin reflecting matrix.

In order to ensure the assumed eigenvector is indeed the eigenvector of the transfer matrix, $\mu_1, \ldots, \mu_n$ should satisfy the following Bethe ansatz equations:

$$\begin{aligned}&\frac{\sin(2\mu_j - \eta)\sin(\mu_j - \tilde{c} + \eta - \tilde{s}\eta)\sin(\mu_j - \tilde{c} + 2\eta + \tilde{s}\eta)}{\sin(2\mu_j + \eta)\sin(\mu_j - \tilde{c} + \eta + \tilde{s}\eta)\sin(\mu_j - \tilde{c} - \tilde{s}\eta)}\sin^{2N}(\mu_j + \eta) \\ &\qquad\times\prod_{i=1}^{n}\sin(\mu_j + \mu_i)\sin(\mu_j - \mu_i - \eta) \\ &\qquad= -\sin^{2N}(\mu_j)\Lambda^{(1)}(\mu_j) \qquad j = 1, 2, \ldots, n.\end{aligned} \tag{73}$$

Here we have used the notation $\Lambda^{(1)}(\lambda)$ to denote the eigenvalue of the nested transfer matrix $t^{(1)}(\lambda)$.

4.2. Bethe ansatz for the six-vertex model with higher spin reflecting matrices

We repeat almost the same procedure as that of the first-level algebraic Bethe ansatz method. We only write down some results without the detailed calculations here. We have

$$\mathrm{e}^{\mathrm{i}\eta}\frac{\sin(2\lambda' - \eta)}{\sin(2\lambda')}D(\lambda', c')|0\rangle_r = -\mathrm{e}^{-\mathrm{i}2\lambda}\frac{\sin(2\lambda)\sin(\lambda + c + \eta - s\eta)}{\sin(2\lambda + \eta)\sin(\lambda - c + s\eta)}|0\rangle_r \equiv U_2|0\rangle \tag{74}$$

$$\mathrm{e}^{\mathrm{i}\eta}\frac{\sin(2\lambda' - \eta)}{\sin(2\lambda')}\left[A(\lambda', c') + D(\lambda', c')\frac{\sin(\eta)\,\mathrm{e}^{-\mathrm{i}2\lambda'}}{\sin(2\lambda' - \eta)}\right]|0\rangle_r \tag{75}$$

$$= -\mathrm{e}^{-\mathrm{i}(2\lambda+\eta)}\frac{\sin(\lambda + c + \eta + s\eta)\sin(\lambda - c - \eta + s\eta)}{\sin(\lambda - c - \eta - s\eta)\sin(\lambda - c + s\eta)}|0\rangle \equiv U_1|0\rangle_r \tag{76}$$

and

$$A^+\left(\lambda' - \frac{\eta}{2}\right)|0\rangle_r = -\mathrm{e}^{\mathrm{i}(2\lambda+\eta)} \frac{\sin(\lambda + \tilde{c} - \eta + \tilde{s}\eta)}{\sin(\lambda - \tilde{c} + \eta + \tilde{s}\eta)}|0\rangle_r \equiv U_1^+|0\rangle_r \tag{77}$$

$$\left[D^+\left(\lambda' - \frac{\eta}{2}\right) - A^+\left(\lambda' - \frac{\eta}{2}\right)\frac{\sin(\eta)\,\mathrm{e}^{-\mathrm{i}2\lambda'}}{\sin(2\lambda' - \eta)}\right]|0\rangle_r$$
$$= -\mathrm{e}^{\mathrm{i}2\lambda} \frac{\sin(2\lambda - \eta)\sin(\lambda + \tilde{c} - \eta - \tilde{s}\eta)\sin(\lambda - \tilde{c} + \eta - \tilde{s}\eta)}{\sin(2\lambda)\sin(\lambda - \tilde{c} - \tilde{s}\eta)\sin(\lambda - \tilde{c} + \eta + \tilde{s}\eta)}|0\rangle_r \equiv U_2^+|0\rangle_r. \tag{78}$$

We then can obtain the reduced double-row monodromy matrix. This double-row monodromy matrix also satisfies the reflection equation with a six-vertex R-matrix. From this reflection equation, we can find the commutation relations. And using the algebraic Bethe ansatz method, we finally obtain the eigenvalues of the nested transfer matrix as

$$\begin{aligned}\Lambda^{(1)}(\lambda') = {}& -U_1^+U_1 \prod_{i=1}^{n}[\sin(\lambda' + \mu_i')\sin(\lambda' - \mu_i')] \\ &\times \prod_{l=1}^{m}\left\{\frac{\sin(\lambda' - \mu_l'^{(1)} - \eta)\sin(\lambda' + \mu_l'^{(1)} - 2\eta)}{\sin(\lambda' - \mu_l'^{(1)})\sin(\lambda' + \mu_l'^{(1)} - \eta)}\right\} \\ & -U_2^+U_2 \prod_{i=1}^{n}[\sin(\lambda' + \mu'_i - \eta)\sin(\lambda' - \mu'_i - \eta)] \\ &\times \prod_{l=1}^{m}\left\{\frac{\sin(\lambda' - \mu_l'^{(1)} + \eta)\sin(\lambda' + \mu_l'^{(1)})}{\sin(\lambda' - \mu_l'^{(1)})\sin(\lambda' + \mu_l'^{(1)} - \eta)}\right\}\end{aligned} \tag{79}$$

where $\mu_1'^{(1)}, \ldots, \mu_m'^{(1)}$ should satisfy the corresponding Bethe ansatz equations. In what follows, we give a summary of our main result.

4.3. Result

The eigenvalues of the transfer matrix for the generalized supersymmetric t–J model are given as follows:

$$\begin{aligned}\Lambda(\lambda) = {}& \frac{\sin(2\lambda - \eta)\sin(\lambda - \tilde{c} + \eta - \tilde{s}\eta)\sin(\lambda - \tilde{c} + 2\eta + \tilde{s}\eta)}{\sin(2\lambda + \eta)\sin(\lambda - \tilde{c} + \eta + \tilde{s}\eta)\sin(\lambda - \tilde{c} - \tilde{s}\eta)} \\ &\times \sin^{2N}(\lambda + \eta)\prod_{i=1}^{n}\frac{\sin(\lambda + \mu_i)\sin(\lambda - \mu_i - \eta)}{\sin(\lambda + \mu_i + \eta)\sin(\lambda - \mu_i)} \\ &+ \sin^{2N}(\lambda)\prod_{i=1}^{n}\frac{1}{\sin(\lambda - \mu_i)\sin(\lambda + \mu_i + \eta)}\Lambda^{(1)}(\lambda)\end{aligned} \tag{80}$$

$$\begin{aligned}\Lambda^{(1)}(\lambda) = {}& -\frac{\sin(\lambda + c + \eta + s\eta)\sin(\lambda - c - \eta + s\eta)\sin(\lambda + \tilde{c} - \eta + \tilde{s}\eta)}{\sin(\lambda - c - \eta - s\eta)\sin(\lambda - c + s\eta)\sin(\lambda - \tilde{c} + \eta + \tilde{s}\eta)} \\ &\times \prod_{i=1}^{n}[\sin(\lambda + \mu_i + \eta)\sin(\lambda - \mu_i)]\prod_{l=1}^{m}\left\{\frac{\sin(\lambda - \mu_l^{(1)} - \eta)\sin(\lambda + \mu_l^{(1)} - \eta)}{\sin(\lambda - \mu_l^{(1)})\sin(\lambda + \mu_l^{(1)})}\right\}\end{aligned}$$

$$-\frac{\sin(2\lambda-\eta)\sin(\lambda+\tilde{c}-\eta-\tilde{s}\eta)\sin(\lambda-\tilde{c}+\eta-\tilde{s}\eta)\sin(\lambda+c+\eta-s\eta)}{\sin(2\lambda+\eta)\sin(\lambda-\tilde{c}-\tilde{s}\eta)\sin(\lambda-\tilde{c}+\eta+\tilde{s}\eta)\sin(\lambda-c+s\eta)}$$

$$\times\prod_{i=1}^{n}[\sin(\lambda+\mu_i)\sin(\lambda-\mu_i-\eta)]\prod_{l=1}^{m}\left\{\frac{\sin(\lambda-\mu_l^{(1)}+\eta)\sin(\lambda+\mu_l^{(1)}+\eta)}{\sin(\lambda-\mu_l^{(1)})\sin(\lambda+\mu_l^{(1)})}\right\} \tag{81}$$

where $\mu_1,\ldots,\mu_n$ and $\mu_1^{(1)},\ldots,\mu_m^{(1)}$ should satisfy the Bethe ansatz equations

$$\frac{\sin(\mu_j^{(1)}+c+\eta+s\eta)\sin(\mu_j^{(1)}-c-\eta+s\eta)\sin(\mu_j^{(1)}+\tilde{c}-\eta+\tilde{s}\eta)\sin(\mu_j^{(1)}-\tilde{c}-\tilde{s}\eta)}{\sin(\mu_j^{(1)}-c-\eta-s\eta)\sin(\mu_j^{(1)}+c+\eta-s\eta)\sin(\mu_j^{(1)}+\tilde{c}-\eta-\tilde{s}\eta)\sin(\mu_j^{(1)}-\tilde{c}+\eta-\tilde{s}\eta)}$$

$$=\prod_{i=1}^{n}\frac{\sin(\mu_j^{(1)}+\mu_i)\sin(\mu_j^{(1)}-\mu_i-\eta)}{\sin(\mu_j^{(1)}+\mu_i+\eta)\sin(\mu_j^{(1)}-\mu_i)}$$

$$\times\prod_{l=1,\neq j}^{m}\frac{\sin(\mu_j^{(1)}-\mu_l^{(1)}+\eta)\sin(\mu_j^{(1)}+\mu_l^{(1)}+\eta)}{\sin(\mu_j^{(1)}-\mu_l^{(1)}-\eta)\sin(\mu_j^{(1)}+\mu_l^{(1)}-\eta)}\qquad j=1,\ldots,m \tag{82}$$

and

$$\frac{\sin(\mu_j+\tilde{c}-\eta-\tilde{s}\eta)\sin(\lambda+c+\eta-s\eta)}{\sin(\mu_j-\tilde{c}+2\eta+\tilde{s}\eta)\sin(\lambda-c+s\eta)}$$

$$=\frac{\sin^{2N}(\mu_j+\eta)}{\sin^{2N}(\mu_j)}\prod_{l=1}^{m}\frac{\sin(\mu_j-\mu_l^{(1)})\sin(\mu_j+\mu_l^{(1)})}{\sin(\mu_j-\mu_l^{(1)}+\eta)\sin(\mu_j+\mu_l^{(1)}+\eta)}\qquad j=1,\ldots,n. \tag{83}$$

With the help of the definition (34), we know the energy of the Hamiltonian (34) takes the following form:

$$E\equiv\left.\frac{\mathrm{d}\ln\Lambda(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=0}$$

$$=(N-2)\cos(\eta)+\sum_{i=1}^{n}\frac{\sin^2(\eta)}{\sin(\mu_i)\sin(\mu_i+\eta)}$$

$$-\sin^2(\eta)\left[\frac{1}{\sin(\tilde{c}-\eta+\tilde{s}\eta)\sin(\tilde{c}+\tilde{s}\eta)}+\frac{1}{\sin(\tilde{c}-\eta-\tilde{s}\eta)\sin(\tilde{c}-2\eta-\tilde{s}\eta)}\right]. \tag{84}$$

5. Summary

In this paper, we have studied the generalized supersymmetric t–J model with boundary impurities. Using the higher spin L operator of the *XXZ* Heisenberg chain and the general diagonal solution to the reflection equation for the six-vertex model, we find a higher spin reflecting matrix for the generalized supersymmetric t–J model. Applying the graded algebraic Bethe ansatz method, we obtain the eigenvalues of the transfer matrix for the t–J model with higher spin boundaries.

It is interesting to solve this problem in other background gradings, for example, FBF or BFF. The higher spin reflecting matrix should be constructed from the BF or FB six-vertex models. The analysis of ground state properties, low-lying excitations and the thermodynamic Bethe ansatz is always worth performing.

One can find that the $SU_q(2)$ higher spin reflecting matrix also satisfies the reflection equation of $SU_q(N)$ model. The eigenvalues of the $SU_q(N)$ model with $SU_q(2)$ higher spin boundary impurities can be obtained by using the nested algebraic Bethe ansatz method. Actually, the $SU_q(2)$ higher spin boundary impurities could be embedded into $SU_q(M|N)$ spin chains with $M \geqslant 2$ or $N \geqslant 2$.

After we put our paper to the cond-mat e-print archive, X Y Ge and H Q Zhou informed us that they have solved a similar problem independently [47], in which the boundary impurities are spin $\frac{1}{2}$.

Acknowledgments

One of the authors, HF is supported by the Japan Society for the Promotion of Science. He would like to thank of Department of Physics, University of Tokyo for hospitality and for the help of Wadati's group. We thank X Y Ge, H Q Zhou and K L Hur for valuable suggestions and comments. We thank B Y Hou, K J Shi, W L Yang and L Zhao for useful discussions. This work was partly supported by the NSFC, Clibming project and NWU Teachers' fund.

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